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GENERALIZED STIRLING NUMBERS OF THE FIRST KIND: MODIFIED APPROACH

B. S. EL-DESOUKY

Department of Mathematics
Faculty of Science
Mansoura University
35516 Mansoura
Egypt
e-mail: b_desouky@yahoo.com

Abstract

A modified approach via differential operator is given to derive a generalization of Stirling numbers of the first kind. This approach combines the two techniques given by Cakic [3] and Blasiak [2]. Some new combinatorial identities and many relations between different types of Stirling numbers are found. Furthermore, some interesting special cases of the generalized Stirling numbers of the first kind are deduced. Finally, a connection between generalized Stirling numbers of the first and second kind is obtained.

1. Introduction

Recently, a modified approach via differential operator to generalized Stirling numbers of the second kind seems more important and attracted the attention of several researchers, see [1-4] and [12]. Many generalizations, extensions, and applications of these numbers are given (see [5-8, 10, 11]). In this article, we give a modified approach via

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differential operator to generalized Stirling numbers of the first kind, denoted by $s(n, k; \bar{r}, \bar{s})$. This approach combines the two techniques given by Cakic [3] and Blasiak [2].

Throughout this article, we use the following conventions and notations:

$$\sum_{k=k_0}^{n} \ldots = 0 \text{ and } \prod_{k=k_0}^{n} \ldots = 1, \text{ for } n < k_0.$$

The generalized falling and rising factorials of x associated with the sequence $\overline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ of order n, where $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are real numbers, are defined, respectively, by

$$(x; \overline{\alpha})_n := \prod_{i=0}^{n-1}(x - \alpha_i), (x; \overline{\alpha})_0 = 1, \text{ and } \langle x; \overline{\alpha}\rangle_n := \prod_{i=0}^{n-1}(x + \alpha_i), \langle x; \overline{\alpha}\rangle_0 = 1.$$

Note that if $\alpha_i = i\lambda,\ i = 0, 1, \ldots, n-1,$ then

$$(x|\lambda)_n := \prod_{i=0}^{n-1}(x - i\lambda), (x|\lambda)_0 = 1, \text{ and } \langle x|\lambda\rangle_n := \prod_{i=0}^{n-1}(x + i\lambda), \langle x|\lambda\rangle_0 = 1.$$

If $\alpha_i = i,\ i = 0, 1, \ldots, n-1,$ the falling and rising factorials are defined, respectively, by

$$(x)_n := \prod_{i=0}^{n-1}(x - i), (x)_0 = 1, \text{ and } \langle x\rangle_n := \prod_{i=0}^{n-1}(x + i), \langle x\rangle_0 = 1.$$

Gould [13] proved that

$$\begin{aligned}(e^x D)^n &= e^{nx}\sum_{k=0}^{n}(-1)^{n-k}s(n, k)D^k \\ &= e^{nx}\sum_{k=1}^{n}s_1(n, k)D^k,\ D := D_x = d/dx, \qquad (1)\end{aligned}$$

where $s(n, k)$ and $s_1(n, k)$ are the usual Stirling numbers and singles Stirling numbers of the first kind, respectively, defined by

$$(x)_n = \sum_{k=0}^{n} s(n, k)x^k, \; s(n, 0) = \delta_{n,0}, \text{ and } s(n, k) = 0, \text{ for } k > n, \tag{2}$$

and

$$\langle x\rangle_n = \sum_{k=0}^{n} s_1(n, k)x^k, \; s_1(n, k) = \delta_{n,0}, \text{ and } s_1(n, k) = 0, \text{ for } k > n. \tag{3}$$

They satisfy, respectively, the recurrence relations

$$s(n+1, k) = s(n, k-1) - ns(n, k), \tag{4}$$

and

$$s_1(n+1, k) = s_1(n, k-1) + ns_1(n, k). \tag{5}$$

Equation (1) is equivalent to

$$(e^{a^+}a)^n = e^{na^+}\sum_{k=1}^{n}(-1)^{n-k}s(n, k)a^k = e^{na^+}\sum_{k=1}^{n}s_1(n, k)a^k, \tag{6}$$

where a^+ and a are boson creation and annihilation operators, respectively.

In Section 2, using the modified differential operator $e^{r_n x}D^{s_n} \ldots e^{r_2 x}D^{s_2}e^{r_1 x}D^{s_1}$, we define the generalized Stirling numbers of the first kind, denoted by $s(k; \bar{r}, \bar{s})$, which are called $(\bar{r}, \bar{s})$-*Stirling numbers*. A recurrence relation and an explicit formula of these numbers are derived. In Section 3, some interesting special cases are discussed. Moreover, some new combinatorial identities are given. In Section 4, a connection between generalized Stirling numbers of the first and second kind is obtained. Finally, a computer program is written using Maple and executed for calculating the generalized Stirling numbers of the first kind and some special cases, see Appendix.

We modify Gould's results [13], see also [9] and [14], and find a generalization of the Stirling numbers of the first kind as follows:

2. The Generalized Stirling Numbers of the First Kind

Let $\bar{r} := (r_1, r_2, \ldots, r_n)$, be a sequence of real numbers and $\bar{s} := (s_1, s_2, \ldots, s_n)$, be a sequence of nonnegative integers.

Definition 1. Let $s(k; \bar{r}, \bar{s})$, called $(\bar{r}, \bar{s})$-*Stirling numbers* of the first kind, be defined by

$$e^{r_n x} D^{s_n} \ldots e^{r_2 x} D^{s_2} e^{r_1 x} D^{s_1} = e^{(\sum_{l=1}^{n} r_l) x} \sum_{k=s_1}^{\beta_n} s(k; \bar{r}, \bar{s}) D^k, \tag{7}$$

where $s_1 \leq k \leq \beta_n$, $s(k; \bar{r}, \bar{s}) = 0$ for $k < s_1$ or $k > \sum_{j=1}^{n} s_j$, $s(0; \bar{r}, \bar{s}) = 1$, and $\beta_n = \sum_{j=1}^{n} s_j$.

Equation (7) is equivalent to

$$e^{r_n a^+} a^{s_n} \cdots e^{r_2 a^+} a^{s_2} e^{r_1 a^+} a^{s_1} = e^{(\sum_{l=1}^{n} r_l) a^+} \sum_{k=s_1}^{\beta_n} s(k; \bar{r}, \bar{s}) a^k,\ s_1 \leq k \leq \beta_n. \tag{8}$$

Theorem 1. *The numbers* $s(k; \bar{r}, \bar{s})$ *satisfy the recurrence relation*

$$s(k; \bar{r} \oplus r_{n+1}, \bar{s} \oplus s_{n+1}) = \sum_{i=0}^{s_{n+1}} \binom{s_{n+1}}{i} (\sum_{j=1}^{n} r_j)^{s_{n+1}-i} s(k-i; \bar{r}, \bar{s}), \tag{9}$$

with the notations $\bar{r} \oplus r_{n+1} := (r_1, r_2, \cdots, r_n, r_{n+1})$ and $\bar{s} \oplus s_{n+1} := (s_1, s_2, \cdots, s_n, s_{n+1})$.

Proof. From (7), we have

$$e^{r_{n+1} x} D^{s_{n+1}} e^{r_n x} D^{s_n} \cdots e^{r_2 x} D^{s_2} e^{r_1 x} D^{s_1}$$

$$= e^{r_{n+1} x} D^{s_{n+1}} (e^{(\sum_{l=1}^{n} r_l) x} \sum_{m=s_1}^{\beta_n} s(m; \bar{r}, \bar{s}) D^m),$$

thus, we obtain

$$e^{(\sum_{l=1}^{n+1} r_l)x} \sum_{k=s_1}^{\beta_{n+1}} s(k;\, \bar{r} \oplus r_{n+1},\, \bar{s} \oplus s_{n+1}) D^k$$

$$= e^{(\sum_{l=1}^{n+1} r_l)x} \sum_{m=s_1}^{\beta_n} s(m;\, \bar{r},\, \bar{s}) (D + \sum_{j=1}^{n} r_j)^{s_{n+1}} D^m$$

$$= e^{(\sum_{l=1}^{n+1} r_l)x} \sum_{m=s_1}^{\beta_n} s(m;\, \bar{r},\, \bar{s}) \sum_{i=0}^{s_{n+1}} \binom{s_{n+1}}{i} (\sum_{j=1}^{n} r_j)^{s_{n+1}-i} D^{m+i}$$

$$= e^{(\sum_{l=1}^{n+1} r_l)x} \sum_{k=s_1}^{\beta_{n+1}} \sum_{i=0}^{s_{n+1}} s(k-i;\, \bar{r},\, \bar{s}) \binom{s_{n+1}}{i} (\sum_{j=1}^{n} r_j)^{s_{n+1}-i} D^k.$$

Equating the coefficients of D^k on both sides yields (9). □

Theorem 2. *The numbers* $s(k;\, \bar{r},\, \bar{s})$ *have the explicit formula*

$$s(k;\, \bar{r},\, \bar{s}) = \sum_{\sigma_{n-1} = \beta_n - k,\, i_l \geq 0} \prod_{l=1}^{n-1} \binom{s_{l+1}}{i_l} (\sum_{j=1}^{l} r_j)^{i_l}, \tag{10}$$

where $\sigma_n = \sum_{j=0}^{n} i_j$, *with* $i_0 = 0$.

Proof. Since

$$(e^{r_2 x} D^{s_2})(e^{r_1 x} D^{s_1}) = e^{(r_1 + r_2)x} (D + r_1 I)^{s_2} D^{s_1}$$

$$= e^{(r_1 + r_2)x} \sum_{i_1=0}^{s_2} \binom{s_2}{i_1} (r_1)^{i_1} D^{s_1 + s_2 - i_1},$$

then

$$(e^{r_3 x} D^{s_3})(e^{r_2 x} D^{s_2})(e^{r_1 x} D^{s_1})$$

$$= (e^{r_3 x} D^{s_3}) e^{(r_1+r_2)x} \sum_{i_1=0}^{s_2} \binom{s_2}{i_1} (r_1)^{i_1} D^{s_1+s_2-i_1}$$

$$= e^{(r_1+r_2+r_3)x} \sum_{i_1=0}^{s_2} \binom{s_2}{i_1} (r_1)^{i_1} [D + (r_1 + r_2) I]^{s_3} D^{s_1+s_2-i_1}$$

$$= e^{(r_1+r_2+r_3)x} \sum_{i_1=0}^{s_2} \sum_{i_2=0}^{s_3} \binom{s_2}{i_1} \binom{s_3}{i_2} (r_1)^{i_1} (r_1 + r_2)^{i_2} D^{s_1+s_2+s_3-i_1-i_2},$$

thus, by iteration, we get

$$e^{r_n x} D^{s_n} \cdots e^{r_2 x} D^{s_2} e^{r_1 x} D^{s_1}$$

$$= e^{(\sum_{l=1}^{n} r_l)x} \sum_{i_1=0}^{s_2} \cdots \sum_{i_{n-1}=0}^{s_n} \prod_{l=1}^{n-1} \binom{s_{l+1}}{i_l} (\sum_{j=1}^{l} r_j)^{i_l} D^{\sum_{j=1}^{n} s_j - \sum_{j=1}^{n-1} i_j}. \tag{11}$$

Setting $\sum_{j=1}^{n} s_j - \sum_{j=1}^{n-1} i_j = \beta_n - \sigma_{n-1} = k$, we obtain

$$e^{r_n x} D^{s_n} \cdots e^{r_2 x} D^{s_2} e^{r_1 x} D^{s_1}$$

$$= e^{(\sum_{l=1}^{n} r_l)x} \sum_{k=s_1}^{\beta_n} \sum_{\sigma_{n-1}=\beta_n-k,\, i_l \ge 0} \prod_{l=1}^{n-1} \binom{s_{l+1}}{i_l} (\sum_{j=1}^{l} r_j)^{i_l} D^k, \tag{12}$$

comparing (7) and (12) yields (10). □

Operating with both sides of (7) on the exponential function e^{lx}, we get

$$l^{s_1} (l + r_1)^{s_2} \cdots (l + r_1 + r_2 + \cdots + r_{n-1})^{s_n} = \sum_{k=s_1}^{\beta_n} s(k;\, \bar{r},\, \bar{s}) l^k.$$

Therefore, since a nonzero polynomial can have only a finite set of zeros, we have

$$\prod_{j=1}^{n} (x + \sum_{i=0}^{j-1} r_i)^{s_j} = \sum_{k=s_1}^{\beta_n} s(k;\, \bar{r},\, \bar{s}) x^k, \text{ where } r_0 = 0. \tag{13}$$

Remark 1. It is worth nothing that, Equation (13) gives a generalization of Comtet numbers [8] of the first kind $s_{\overline{\alpha}}(n, k)$ associated with the sequence $\overline{\alpha} = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$, defined by

$$(t; \overline{\alpha})_n = \sum_{k=0}^{n} s_{\overline{\alpha}}(n, k)t^k, \text{ where } \alpha_{j-1} = \sum_{i=0}^{j-1} - r_i, \ j = 1, \cdots, n. \tag{14}$$

Setting $x = 1$ in (13), we have the identity

$$\sum_{k=s_1}^{\beta_n} s(k; \overline{r}, \overline{s}) = \prod_{j=1}^{n} (1 + \sum_{i=0}^{j-1} r_i)^{s_j}. \tag{15}$$

Next, we derive some interesting special cases.

3. Special Cases

Setting $r_i = r$ and $s_i = s, i = 1, \cdots, n$ in (7), then we have the following definition:

Definition 2. For any real number r and nonnegative integer s, let the numbers $s(n, k; r, s)$, called (r, s)- *Stirling numbers* of the first kind, be defined by

$$(e^{rx} D^s)^n = e^{nrx} \sum_{k=s}^{ns} s(n, k; r, s) D^k, \tag{16}$$

where $s(0, 0; r, s) = 1$ and $s(n, k; r, s) = 0$ for $s > k > ns$.

Corollary 1. *The numbers* $s(n, k; r, s)$ *satisfy the recurrence relation*

$$s(n+1, k; r, s) = \sum_{i=0}^{s} \binom{s}{i} (nr)^{s-i} s(n, k-i; r, s). \tag{17}$$

Proof. The proof follows directly from (9) by setting $r_i = r$ and $s_i = s, i = 1, \cdots, n.$ $\square$

Corollary 2. *The numbers* $s(n, k; r, s)$ *have the explicit formula*

$$s(n, k; r, s) = r^{ns-k} \sum_{\sigma_{n-1}=ns-k, i_j \geq 0} \prod_{j=1}^{n-1} \binom{s}{i_j} j^{i_j}, \; i_0 = 0. \tag{18}$$

Proof. The proof follows by setting $r_i = r$ and $s_i = s$, $i = 1, \cdots, n$ in (10). □

Setting $r_i = r$ and $s_i = s$, $i = 1, \cdots, n$ in (13), we have

$$\prod_{i=0}^{n-1} (x + ir)^s = (\langle x|r \rangle_n)^s = \sum_{k=s}^{ns} s(n, k; r, s) x^k. \tag{19}$$

Moreover, we discuss the following two special cases:

(i) Setting $r = 1$ in (16), then the numbers $s(n, k; 1, s)$, are defined by

$$(e^x D^s) = e^{nx} \sum_{k=s}^{ns} s(n, k; 1, s) x^k D^k, \tag{20}$$

where $s(n, k; 1, s) = 0$ for $s > k > ns$ and $s(0, 0; 1, s) = 1$.

Corollary 3. *The numbers* $s(n, k; 1, s)$, *satisfy the recurrence relation*

$$s(n+1, k; 1, s) = \sum_{i=0}^{s} \binom{s}{i} n^{s-i} s(n, k-i; 1, s), \tag{21}$$

and have the explicit formula

$$s(n, k; 1, s) = \sum_{\sigma_{n-1}=ns-k, i_j \geq 0} \left(\prod_{j=1}^{n-1} \binom{s}{i_j} j^{i_j} \right), \; \text{where } i_0 = 0. \tag{22}$$

Proof. The proof follows directly from (17) and (18), respectively, by setting $r = 1$. □

Remark 2. In fact, Equations (18) and (22) show that

$$s(n, k; r, s) = r^{ns-k} s(n, k; 1, s). \tag{23}$$

Setting $r = 1$ in (19), we have

$$\Big(\prod_{i=0}^{n-1}(x+i)\Big)^s = (\langle x\rangle_n)^s = \sum_{k=s_1}^{ns} s(n, k; 1, s)x^k. \tag{24}$$

Corollary 4. *The numbers* $s(n, k; 1, 1) := s_1(n, k)$ *satisfy the recurrence relation* (5) *and have the new explicit formula*

$$s_1(n, k) = \sum_{\sigma_{n-1}=n-k,\, i_j\in\{0,1\}} \prod_{j=1}^{n-1} j^{i_j}. \tag{25}$$

Proof. The proof follows by setting $s = 1$ in (21) and (22), respectively. □

Thus, from (25) and [7, Theorem 8.1, p. 280], we have the identity

$$\sum_{\sigma_{n-1}=n-k,\, i_j\in\{0,1\}} \prod_{j=1}^{n-1} j^{i_j} = \sum i_1 i_2 \cdots i_{n-k}, \tag{26}$$

where the summation, on the right hand side, is extended over all $(n-k)$- combinations $\{i_1, i_2, \cdots, i_{n-k}\}$ of the $(n-1)$ positive integers $\{1, 2, \cdots, n-1\}$.

(ii) Setting $s = 1$ in (16), then the numbers $s(n, k; r, 1)$ are defined by

$$(e^{rx}D)^n = e^{rnx}\sum_{k=1}^{n} s(n, k; r, 1)D^k, \tag{27}$$

where $s(n, 0; r, 1) = \delta_{n,0}$, $s(n, k; r, 1) = 0$ for $k > n$.

Corollary 5. *The numbers* $s(n, k; r, 1)$ *satisfy the recurrence relation*

$$s(n+1, k; r, 1) = s(n, k-1; r, 1) + nr\, s(n, k; r, 1). \tag{28}$$

Proof. The proof follows directly by setting $s = 1$ in (17). □

Theorem 3. *The numbers* $s(n, k; r, 1)$ *have the exponential generating function*

$$f_k(t;\ r) = \sum_{n=k}^{\infty} s(n,\ k;\ r,\ 1)\frac{t^n}{n!} = \frac{1}{r^k k!}(-\log(1 - rt))^k. \tag{29}$$

Proof. The proof is left. □

Setting $s = 1$ in (19), we have

$$\prod_{i=0}^{n-1}(x + ir) = \langle x|r\rangle_n = \sum_{k=1}^{n} s(n,\ k;\ r,\ 1)x^k. \tag{30}$$

Thus, if putting $x = 1$ in (30), we have

$$\sum_{k=1}^{n} s(n,\ k;\ r,\ 1) = \prod_{i=0}^{n-1}(1 + ir), \tag{31}$$

hence putting $r = 1,$ yields the well known identity

$$\sum_{k=1}^{n} s(n,\ k;\ 1,\ 1) = \sum_{k=1}^{n} s_1(n,\ k) = \prod_{i=0}^{n-1}(1 + i) = n!. \tag{32}$$

From (19) and (30), then using Cauchy rule of multiplication of series, we get

$$\sum_{k=s}^{ns} s(n,\ k;\ r,\ s)x^k = \left(\sum_{k=1}^{n} s(n,\ k;\ r,\ 1)x^k\right)^s = \prod_{i=1}^{s}\sum_{k_i=1}^{n} s(n,\ k_i;\ r,\ 1)x^{k_i}$$

$$= \sum_{k=s}^{ns}\ \sum_{k_1+k_2+\cdots+k_s=k,\, k_i\geq 1}\ \prod_{i=1}^{s} s(n,\ k_i;\ r,\ 1)x^k,$$

hence

$$s(n,\ k;\ r,\ s) = \sum_{k_1+k_2+\cdots+k_s=k,\, k_i\geq 1}\ \prod_{i=1}^{s} s(n,\ k_i;\ r,\ 1). \tag{33}$$

For example, if $n = 5,\ r = 2,$ and $s = 2,$ then for $k = 2,$ we have $s(5,\ 2;\ 2,\ 2) = 147456,$ Table 4, and $\sum_{k_1+k_2=2,\, k_i\geq 1}\prod_{i=1}^{2} s(5,\ k_i;\ 2,\ 1) =$ $s(5,\ 1;\ 2,\ 1)s(5,\ 1;\ 2,\ 1) = (384)^2 = 147456.$ (Table 2)

This shows that $s(n, k; r, s)$ can be represented in terms of $(r, 1)$-*Stirling numbers*.

Since $s(n, k_i; r, 1) = r^{n-k_i} s_1(n, k_i)$, hence (33) gives

$$s(n, k; r, s) = \sum_{k_1+k_2+\cdots+k_s=k,\, k_i \geq 1} \prod_{i=1}^{s} r^{n-k_i} s_1(n, k_i)$$

$$= r^{ns-k} \sum_{k_1+k_2+\cdots+k_s=k,\, k_i \geq 1} \prod_{i=1}^{s} s_1(n, k_i). \tag{34}$$

This shows that $s(n, k; r, s)$ can be represented in terms of the singles Stirling numbers of the first kind. For example, if $n = 5, r = 2,$ and $s = 3,$ then for $k = 12,$ we get $s(5, 12; 2, 3) = 26000,$ Table 5, and

$$2^{15-12} \sum_{k_1+k_2+k_3=12,\, k_i \geq 1} \prod_{i=1}^{3} s_1(5, k_i) = 2^3(6s_1(5, 3)s_1(5, 4)s_1(5, 5)$$

$$+3s_1(5, 2)s_1(5, 5)s_1(5, 5) + s_1(5, 4)s_1(5, 4)s_1(5, 4))$$

$$= 2^3\left(6(350) + 3(50) + (10)^3\right) = 8(3250) = 26000. \text{ (Table 1)}$$

Using (3) and Cauchy rule of multiplication of series, we get

$$(\langle x \rangle_n)^s = (\sum_{k=1}^{n} s_1(n, k_i)x^{k_i}) = \sum_{k=s}^{ns} \sum_{k_1+k_2+\cdots+k_s=k,\, k_i \geq 1} \prod_{i=1}^{s} s_1(n, k_i)x^k,$$

hence, by virtue of (24), we obtain

$$s(n, k; 1, s) = \sum_{k_1+k_2+\cdots+k_s=k,\, k_i \geq 1} \prod_{i=1}^{s} s_1(n, k_i). \tag{35}$$

Similarly, $s(n, k; 1, s)$ can be represented in terms of the singles Stirling numbers of the first kind. For example, if $n = 5, r = 1,$ and $s = 2,$ then for $k = 2$, we have $s(5,\ 2;\ 1,\ 2) = 576$, Table 3, and $\sum_{k_1+k_2=2,\ k_i\geq 1}\prod_{i=1}^{2}$ $s(5,\ k_i) = s(5,\ 1)s(5,\ 1) = (24)^2 = 576$. (Table 1)

From (22) and (35), we have the interesting identity

$$\sum_{\sigma_n=ns-k,\ i_j\geq 0}\left(\prod_{j=1}^{n-1}\binom{s}{i_j}j^{i_j}\right) = \sum_{k_1+k_2+\cdots+k_s=k,\ k_i\geq 1}\ \prod_{i=1}^{s} s_1(n,\ k_i). \tag{36}$$

4. Connection Between Generalized Stirling Numbers of the First and Second Kind

Setting $e^x = t$, we have $D = t\dfrac{d}{dt} = tD_t$, then substituting in (7), it becomes

$$(t^{r_n}(tD_t)^{s_n})\cdots(t^{r_2}(tD_t)^{s_2})(t^{r_1}(tD_t)^{s_1}) = t^{\sum_{i=1}^{n} r_i}\sum_{k=s_1}^{\beta_n} s(n,\ k;\ r,\ s)(tD_t)^k. \tag{37}$$

The LHS of (37), for $n = 1$, gives

$$(t^{r_1}(tD_t)^{s_1}) = t^{r_1}\sum_{k_1=0}^{s_1} S(s_1,\ k_1)t^{k_1}(D_t)^{k_1}$$

$$= \sum_{k_1=0}^{s_1} S(s_1,\ k_1)t^{r_1+k_1}(D_t)^{k_1}.$$

For $n = 2$,

$$(t^{r_2}(tD_t)^{s_2})(t^{r_1}(tD_t)^{s_1})$$

$$= t^{r_2}\sum_{k_2=0}^{s_2} S(s_2,\ k_2)t^{k_2}(D_t)^{k_2}\sum_{k_1=0}^{s_1} S(s_1,\ k_1)t^{r_1+k_1}(D_t)^{k_1}$$

$$= \sum_{k_2=0}^{s_2}\sum_{k_1=0}^{s_1} S(s_2,\ k_2)S(s_1,\ k_1)t^{r_2+k_2}(D_t)^{k_2}t^{r_1+k_1}(D_t)^{k_1}.$$

Successively, we obtain

$$(t^{r_n}(tD_t)^{s_n} \ldots t^{r_2}(tD_t)^{s_2} t^{r_1}(tD_t)^{s_1})$$

$$= \sum_{k_n=0}^{s_n} \cdots \sum_{k_2=0}^{s_2} \sum_{k_1=0}^{s_1} \prod_{i=1}^{n} S(s_i, k_i) t^{r_n+k_n}(D_t)^{k_n} \ldots t^{r_2+k_2}(D_t)^{k_2} t^{r_1+k_1}(D_t)^{k_1}$$

$$= \sum_{k_n=0}^{s_n} \cdots \sum_{k_2=0}^{s_2} \sum_{k_1=0}^{s_1} \prod_{i=1}^{n} S(s_i, k_i) t^{\rho_n}(D_t)^{k_n} \ldots t^{\rho_2}(D_t)^{k_2} t^{\rho_1}(D_t)^{k_1},$$

where $\overline{\rho} = (\rho_1, \rho_2, \cdots, \rho_n) = \overline{r} + \overline{k}$, i.e., $\rho_i = r_i + k_i$, $i = 1, 2 \cdots, n$.

From [12, Equation (5)], see also [2], we have

$$t^{\rho_n}(D_t)^{k_n} \ldots t^{\rho_2}(D_t)^{k_2} t^{\rho_1}(D_t)^{k_1} = t_n^d \sum_{\ell=k_1}^{k_1+\cdots+k_n} S_{\overline{\rho}, \overline{k}}(\ell) t^{\ell} D_t^{\ell}, \tag{38}$$

where $d_n = r_1 + r_2 + \cdots + r_n$. Thus

$$t^{r_n}(tD_t)^{s_n} \ldots t^{r_2}(tD_t)^{s_2} t^{r_1}(tD_t)^{s_1}$$

$$= \sum_{k_n=0}^{s_n} \cdots \sum_{k_2=0}^{s_2} \sum_{k_1=0}^{s_1} \sum_{\ell=k_1}^{k_1+k_2+\cdots+k_n} \prod_{i=1}^{n} S(s_i, k_i) t^{\sum_{l=0}^{n} r_l} \sum_{\ell=k_1}^{k_1+k_2+\cdots+k_n} s_{\overline{\rho}, \overline{k}}(\ell) t^{\ell} D_t^{\ell}$$

$$= \sum_{j=0}^{\beta_n} \sum_{k_1+k_2+\cdots+k_n=j} \sum_{\ell=k_1}^{k_1+k_2+\cdots+k_n} t^{\sum_{l=0}^{n} r_l} \prod_{i=1}^{n} S(s_i, k_i) S_{\overline{\rho}, \overline{k}}(\ell) t^{\ell} D_t^{\ell},$$

hence

$$t^{r_n}(tD_t)^{s_n} \ldots t^{r_2}(tD_t)^{s_2} t^{r_1}(tD_t)^{s_1}$$

$$= t^{\sum_{l=0}^{n} r_l} \sum_{\ell=0}^{\beta_n} \sum_{j=\ell}^{\beta_n} \sum_{k_1+k_2+\cdots+k_n=j} \prod_{i=1}^{n} S(s_i, k_i) S_{\overline{\rho}, \overline{k}}(\ell) t^{\ell} D_t^{\ell}. \tag{39}$$

From the RHS of (37), we get

$$t^{\sum_{l=0}^{n} r_l} \sum_{k=s_1}^{\beta_n} (n, k; \bar{r}, \bar{s})(tD_t)^k = t^{\sum_{l=0}^{n} r_l} \sum_{k=s_1}^{\beta_n} s(n, k; \bar{r}, \bar{s}) \sum_{\ell=0}^{k} S(k, \ell) t^{\ell} D_t^{\ell}$$

$$= t^{\sum_{l=0}^{n} r_l} \sum_{\ell=0}^{\beta_n} \sum_{k=\ell}^{\beta_n} s(n, k; \bar{r}, \bar{s}) S(k, \ell) t^{\ell} D_t^{\ell}. \tag{40}$$

Equating of coefficients of $t^{\ell} D^{\ell}$ in (39) and (40), we obtain the interesting identity

$$\sum_{j=\ell}^{\beta_n} \sum_{k_1+k_2+\cdots+k_n=j} \prod_{i=1}^{n} S(s_i, k_i) S_{\bar{\rho}, \bar{k}}(\ell) = \sum_{k=\ell}^{\beta_n} s(n, k; \bar{r}, \bar{s}) S(k, \ell). \tag{41}$$

This identity gives a connection between generalized Stirling numbers of the first and second kind.

Setting $r_i = r$ and $s_i = s$ in (40), we get

$$\sum_{j=\ell}^{ns} \sum_{k_1+k_2+\cdots+k_n=j} \prod_{i=1}^{n} S(s_i, k_i) S_{\bar{\rho}, \bar{k}}(n, \ell) = \sum_{j=\ell}^{ns} s(n, k; r, s) S(k, \ell). \tag{42}$$

For example, if $n = 2, s = 2, r = 1,$ and $\ell = 2,$ we have

$$\sum_{j=2}^{4} \sum_{k_1+k_2=j} \prod_{i=1}^{2} S(2, k_i) S_{(1+k_1, 1+k_2), (k_1, k_2)}(2, 2) = \sum_{k=2}^{4} s(2, k; 2, 2) S(k, 2). \tag{43}$$

Using [12, Equation (10)] and Table 1, we get

$$\text{LHS of (43)} = S(2, 1)S(2, 1)S_{(2,2),(1,1)}(2, 2) + S(2, 1)S(2, 2)S_{(2,3),(1,2)}(2, 2)$$

$$+ S(2, 2)S(2, 1)S_{(3,2),(2,1)}(2, 2) + S(2, 2)S(2, 2)S_{(3,3),(2,2)}(2, 2) = 1(1)(1)$$

$$+ 1(1)(4) + 1(1)(3) + 1(1)(6) = 14.$$

$$\text{RHS of (43)} = \sum_{k=2}^{4} s(2, k; 1, 2)S(k, 2) = s(2, 2; 1, 2)S(2, 2) + s(2, 3; 1, 2)S(3, 2)$$

$$+ s(2, 4; 1, 2)S(4, 2) = 1(1) + 2(3) + 1(7) = 14. \text{ (Tables 1 and 3)}$$

This confirms (43).

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Appendix

Table 1. $n = 5,\ r = 1,\ s = 1,\ 1 \le k \le 5$

1	0	0	0	0	1
1	1	0	0	0	2
2	3	1	0	0	6
6	11	6	1	0	24
24	50	35	10	1	120

Table 2. $n = 5,\ r = 2,\ s = 1,\ 1 \le k \le 5$

1	0	0	0	0	1
2	1	0	0	0	3
8	6	1	0	0	15
48	44	12	1	0	105
384	400	140	20	1	945

Table 3. $n = 5,\ r = 1,\ s = 2,\ 2 \le k \le 10$

1	0	0	0	0	0	0	0	0	1
1	2	1	0	0	0	0	0	0	4
4	12	13	6	1	0	0	0	0	36
36	132	193	144	58	12	1	0	0	576
576	2400	4180	3980	2273	800	170	20	1	14400

Table 4. $n = 5,\ r = 2,\ s = 2,\ 2 \le k \le 10$

1	0	0	0	0	0	0	0	0	1
4	4	1	0	0	0	0	0	0	9
64	96	52	12	1	0	0	0	0	225
2304	4224	3088	1152	232	24	1	0	0	11025
147456	307200	267520	127360	36368	6400	680	40	1	893025

■

Table 5. $n = 5,\ r = 2,\ s = 3,\ 3 \leq k \leq 15$

1	0	0	0	0	0	0	0	0	0	0	0	0	1
8	12	6	1	0	0	0	0	0	0	0	0	0	27
512	1152	1056	504	132	18	1	0	0	0	0	0	0	3375
110592	304128	361728	244160	103104	28272	5040	564	36	1	0	0	0	1157625
56623104	176947200	246251520	201871360	108653568	40492800	10727360	2038080	275952	26000	1620	60	1	843908625